THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2060B Mathematical Analysis II (Spring 2017)
HW5 Solution

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1. (P.215 Q11)

We first show that $f \in R[a,b]$: Since f is bounded, by Prop. 1.8 of the Lecture note, it suffices to show that for all $\epsilon > 0$, there exists a partition $P := a = x_0 < x_1 < ... < x_n = b$ on $[a,b]$, we have

$$U(f,P) - L(f,P) < \epsilon$$

Let $\epsilon > 0$ be given, choose $c = a + \delta$, where $0 < \delta < \min\{\frac{\epsilon}{4M+1}, b-a\}$

Then $c \in (a,b)$, and hence by the integrability of f on $[c,b]$, there exists a partition $P' := c = x_0 < x_1 < ... < x_n = b$ on $[c,b]$ such that

$$U(f,P') - L(f,P') < \frac{\epsilon}{2}$$

Define a partition P on $[a,b]$ by $P := a < c < x_1 < ... < x_n = b$. Then

$$\begin{aligned} U(f,P) - L(f,P) &= (\sup_{[a,c]} f - \inf_{[a,c]} f)(c-a) + U(f,P') - L(f,P') \\ &< 2M \cdot \frac{\epsilon}{4M+1} + \frac{\epsilon}{2} \\ &< \epsilon \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $f \in R[a,b]$.

Then we claim that $\int_c^b f \to \int_a^b f$ as $c \to a^-$: Given $\epsilon > 0$, choose $\delta = \min\{\frac{\epsilon}{M+1}, b-a\}$. Then for all $a < c < a + \delta$, since $f \in R[a,b]$ and $f|_{[c,b]} \in R[c,b]$, by Prop. 1.13 of the note,

$$|\int_c^b f - \int_a^b f| = |\int_a^c f|$$

By Prop. 1.12 (ii), $|\int_a^c f| \le \int_a^c |f| \le M(c-a) < M \cdot \frac{\epsilon}{M+1} < \epsilon$
Therefore, for all $a < c < a + \delta$, $|\int_c^b f - \int_a^b f| < \epsilon$. This shows $\int_c^b f \to \int_a^b f$ as $c \to a^-$.

2. (P.215 Q15)

Note that an analogous argument as in Q11 implies that for any bounded function $f : [a,b] \to \mathbb{R}$ such that for any $a < c < b$, $f|_{[a,c]} \in R[a,c]$, then $f \in R[a,b]$.

More generally, the argument will actually imply that for any bounded function $f : [a,b] \to \mathbb{R}$ such that for any $a < c < d < b$, $f|_{[c,d]} \in R[c,d]$, then $f \in R[a,b]$.

Now let $E = \{y_1, ..., y_N\} \subseteq [a,b]$ be the given finite set such that $y_1 < y_2 < ... < y_N$. We first assume for simplicity that $y_1 \neq a$ and $y_N \neq b$. Denote $y_0 = a$ and $y_{N+1} = b$ for notational convenience.

By Prop. 1.13, and induction on N, it suffices to show that $f_0 = f|_{[y_0,y_1]}, f_1 = f|_{[y_1,y_2]},...,f_{N-1} = f|_{[y_{N-1},y_N]}, f_N = f|_{[y_N,y_{N+1}]}$ are integrable on their corresponding domains:

For each $0 \leq k \leq N$, for all $c, d \in \mathbb{R}$ such that $y_k < c < d < y_{k+1}$, since$f$ is continuous on $[a,b]\backslash E$, $f_k|_{[c,d]}$ is continuous and hence $f_k|_{[c,d]} \in R[c,d]$. By the second assertion in the above, $f_k \in R[y_k, y_{k+1}]$.

Therefore, for each $0 \leq k \leq N$,$f_k \in R[y_k, y_{k+1}]$, and hence $f \in R[a,b]$.

If $y_1 = a$ (resp. $y_N = b$), simply disregard f_0 (resp. f_N) and the above argument still applies.

3. (P.215 Q16)

   Define $F(x) = \begin{cases} 0 & \text{if } x = a \\ \int_a^x f & \text{if } a < x \leq b \end{cases}$

   Then by Theorem 2.1 (ii) of the lecture note, since f is continuous on $[a,b]$, F is continuous on $[a,b]$, differentiable on (a,b) with $F' = f$ on (a,b). Therefore, by Mean Value Theorem (Theorem 6.2.4 of the textbook), there exists $c \in (a,b)$ such that
   $$F(b) - F(a) = F'(c)(b-a)$$
   which is exactly the following equality:
   $$\int_a^b f - 0 = f(c)(b-a)$$
   Therefore, there exists $c \in (a,b)$ such that $\int_a^b f = f(c)(b-a)$.

4. Define $F(x) = \begin{cases} 0 & \text{if } x = a \\ \int_a^x fg & \text{if } a < x \leq b \end{cases}$ and $G(x) = \begin{cases} 0 & \text{if } x = a \\ \int_a^x g & \text{if } a < x \leq b \end{cases}$.

   Again, by Theorem 2.1, F, G are continuous on $[a,b]$, differentiable on (a,b) with $F' = fg$; $G' = g$ on (a,b). Since $g(x) > 0$ for all $x \in [a,b]$, $G'(x) \neq 0$ for all $x \in (a,b)$. Therefore, by Cauchy Mean Value Theorem (Theorem 6.3.2 of the textbook), there exists $c \in (a,b)$ such that
   $$\frac{F(b) - F(a)}{G(b) - G(a)} = \frac{F'(c)}{G'(c)}$$
   which is exactly the following equality:
   $$\frac{\int_a^b fg - 0}{\int_a^b g - 0} = \frac{(fg)(c)}{g(c)} = f(c)$$
   Therefore, there exists $c \in (a,b)$ such that
   $$\int_a^b fg = f(c) \int_a^b g$$

   This conclusion fails without the assumption that $g(x) > 0$ for all $x \in [a,b]$. For example, let $a = -1$; $b = 1$; $f(x) = g(x) = x$. Then $\int_{-1}^1 g(x)dx = 0$, and hence for all $c \in [-1,1]$, $f(c) \int_{-1}^1 g = 0$. Meanwhile, $\int_{-1}^1 (fg)(x)dx = 2 \int_0^1 x^2 dx = \dfrac{2}{3} \neq 0$. Therefore, the conclusion fails.